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ON SUBMANIFOLDS OF HOPF MANIFOLDS

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To Professor A. Cossu on the Occasion of his 65th Birthday

ABSTRACT
The main results we obtain are as follows: an invariant submanifold of a Hopf manifold with semi-flat normal connection is either a complex hypersurface or a totally-umbilical quasi-Einstein submanifold with a flat normal connection. The only totally-umbilical invariant submanifolds of zero scalar curvature of a Hopf manifold are the totally-geodesic flat surfaces.

1. Introduction

Let H^m be a Hopf manifold of complex dimension m, $m > 2$ (cf. [10, p. 137]); it is a PK_0-manifold (in the sense of [12]) with the Hermitian metric g_0 given by the diffeomorphism $H^m \approx S^{2m-1} \times S^1$, cf. [12, p. 264]. Let J denote the complex structure on H^m. Let M be a submanifold of H^m of real dimension $2n$, $n \leqq m$. We denote by $E \to M$ the normal bundle of the given immersion of M in H^m. Let $\nabla^\perp$ be the normal connection (in the normal bundle E) of M, and $R^\perp$ its curvature. Let g denote the first fundamental form; we say M has *semi-flat* normal connection if

$$R^\perp(X, Y)\xi = \rho_0 g(X, PY)\mathrm{nor}(J\xi) \tag{1.1}$$

for some real-valued smooth function ρ_0 on M and any tangent vector fields X, Y on M, respectively any (normal) section ξ in E. This generalizes slightly the definition in [15, p. 107] where $\rho_0 = 2$; actually, if M is invariant (i.e. $J_x(T_x(M)) = T_x(M)$, for any $x \in M$) and the ambient space is Kaehler and $n \geqq 2$, then $\rho_0 =$ constant, cf. lemma 9.1 in [15, p. 114]. Also $PX = \tan(JX)$,

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while $\tan_x$, nor_x denote the natural projections of the direct sum decomposition $T_x(H^m) = T_x(M) \oplus E_x$, $x \in M$. We obtain the following:

THEOREM 1. *Let M be an invariant submanifold with a semi-flat normal connection of the Hopf manifold H^m, $\dim(M) = 2n$. If $m - n > 1$ then M is a totally-umbilical quasi-Einstein submanifold with a flat normal connection.*

To clarify Theorem 1, we also state:

THEOREM 2. *Any totally-umbilical submanifold of a Hopf manifold is a quasi-Einstein manifold.*

Let Ric be the Ricci form of M; then M is a *quasi-Einstein* submanifold (cf. [6]) if $\mathrm{Ric} = a \cdot g + b \cdot \omega \otimes \omega$, for some real-valued smooth functions a, b on M. Here ω denotes the 1-form naturally induced on M by the Lee form of H^m. If M is invariant then (M, g) is a locally conformal Kaehler (l.c.K.) manifold (cf. [12–14]) and

$$\omega = \frac{1}{n-1}\, i(\Omega)d\Omega,$$

where $i(\quad\cdot\quad)$ is the interior product (cf. [5]) and Ω the Kaehler 2-form of M. We also obtain:

THEOREM 3. *Let M be an invariant submanifold with semi-flat normal connection in H^m.*

(i) *If* $\mathrm{codim}(M) = 2$, $n \geqq 2$, *and* (1.1) *holds for some smooth function* $\rho_0 : M \to (0, +\infty)$, *then M is a globally conformal Kaehler manifold and its Lee form is given by* $\omega = -d \log \rho_0$.

(ii) *If* $\mathrm{codim}(M) > 2$ *and M is strongly non-Kaehler then M cannot be an Einstein manifold.*

Invariant submanifolds of l.c.K. manifolds were considered in [2], [14] and recently in [8].

We shall need the *Yamabe functional* of a compact submanifold M:

$$I(\varphi) = \|\varphi\|_N^{-2} E(\varphi) \tag{1.2}$$

where

$$E(\varphi) = \int_M (a\,\|d\varphi\|^2 + \rho\varphi^2) * 1,$$

$$\| \varphi \|_N = \left(\int_M |\varphi|^N * 1 \right)^{1/N}, \qquad N = \frac{2n}{n-1}, \qquad a = \frac{2(2n-1)}{n-1},$$

and ρ denotes the scalar curvature of M; cf. J. M. Lee and T. H. Parker [11]. The *Yamabe invariant* (a conformal invariant) of M is given by:

$$\mu_0 = \inf\{I(\varphi) : \varphi \in C^\infty(M), \varphi \geqq 0, \varphi \not\equiv 0\}$$

where $C^\infty(M)$ denotes the ring of all real-valued smooth functions on M. We obtain:

THEOREM 4. *Let M be a compact totally-umbilical submanifold of real dimension $2n$ of the Hopf manifold H^m. If $n \geqq \frac{1}{4} \| \omega \|^2$ and M has a non-positive Yamabe invariant, i.e. $\mu_0 \leqq 0$, then M is totally-geodesic. Moreover, if additionally M is invariant then it is a generalized Hopf manifold, and it is a globally conformal Kaehler manifold provided that the sectional curvatures of M are subject to $k(p) \geqq 1 - \frac{1}{2}(\omega(X)^2 + \omega(Y)^2)$, for any $p \in G_2(M)$ and any g-orthonormal basis $\{X, Y\}$ in p.*

Here $G_2(M) \to M$ denotes the Grassmann bundle of all 2-planes on M. The rest of the paper is devoted to the study of totally-umbilical submanifolds (of Hopf manifolds) with further restrictions on curvature. We obtain:

THEOREM 5. *Let M^{2n} be a real $2n$-dimensional totally-umbilical submanifold of H^m, $2 \leqq n \leqq m$; if M^{2n} is conformally-flat then it has a vanishing scalar curvature.*

By a theorem of S. Goldberg and M. Okumura [7], if M^{2n} is conformally-flat and has constant scalar curvature ρ and the length of the Ricci tensor is $\leqq \rho(2n-1)^{-1/2}$, $n \geqq 2$, then M^{2n} is a space-form. If in turn M^{2n} is a totally-umbilical submanifold of H^m, then we have $\| \mathrm{Ric} \| = 0$ if and only if M^{2n} is a surface, i.e. $n = 1$. We actually prove:

THEOREM 6. *Let M^{2n}, $n \geqq 2$, be a conformally-flat totally-umbilical submanifold of H^m. Then M^{2n} is never a space of constant curvature.*

THEOREM 7. *The only invariant totally-umbilical submanifolds of zero scalar curvature in a Hopf manifold are the totally-geodesic flat surfaces.*

THEOREM 8. *In a Hopf manifold there do not exist any projectively-flat totally umbilical submanifolds with zero scalar curvature and nowhere vanishing 1-form ω.*

The author is grateful to the referee for pointing out to him that actually Theorems 1 to 8 hold for the more general case of an ambient PK_0-manifold, as well as for several suggestions which improved the original form of the manuscript.

2. Basic formulae

Let ω^* be the Lee form of the Hopf manifold (H^m, g_0). Then $B^* = \#\omega^*$ is the Lee field, and # denotes raising of indices with respect to g_0. Let ∇^* be the Levi–Civita connection of g_0; since g_0 is only a l.c.K. metric, ∇^* is not almost-complex. Yet H^m admits a significant almost-complex connection, namely the Weyl connection:

$$D^*_X Y = \nabla^*_X Y - \tfrac{1}{2}(\omega^*(X)Y + \omega^*(Y)X - g_0(X, Y)B^*) \tag{2.1}$$

for any $X, Y \in \mathscr{X}(H^m)$. In general, if M is a manifold then $\mathscr{M}(M)$ denotes the module of all tangent vector fields on M over the ring $C^\infty(M)$. Also, if $F \to M$ is a vector bundle over M then $\Gamma(F)$ denotes the $C^\infty(M)$-module of all smooth cross-setions in F, and F_x denotes the fibre over $x \in M$ in F.

We recall that the Weyl connection of H^m is flat while the curvature R^* of ∇^* is furnished by

$$\begin{aligned} R^*(X, Y)Z = \tfrac{1}{4}\{&[\omega^*(X)Y - \omega^*(Y)X]\omega^*(Z) \\ &+ [g_0(X, Z)\omega^*(Y) - g_0(Y, Z)\omega^*(X)]B^*\} \\ &+ g_0(Y, Z)X - g_0(X, Z)Y \end{aligned} \tag{2.2}$$

for any $X, Y, Z \in \mathscr{X}(H^m)$. See [4]. If M is a submanifold of H^m we denote by h, ∇, a_ξ the second fundamental form, the induced connection and the Weingarten operator (corresponding to the normal section $\xi \in \Gamma(E)$). These satisfy the Gauss and Weingarten equations:

$$\nabla^*_X Y = \nabla_X Y + h(X, Y), \qquad \nabla^*_X \xi = -a_\xi X + \nabla^\perp_X \xi \tag{2.3}$$

for $X, Y \in \mathscr{X}(M)$, $\xi \in \Gamma(E)$. Next we define $D_X Y = \tan(D^*_X Y)$, $A_\xi X = -\tan(D^*_X \xi)$, $H(X, Y) = \operatorname{nor}(D^*_X Y)$, $D^\perp_X \xi = \operatorname{nor}(D^*_X \xi)$, for all $X, Y \in \mathscr{X}(M)$, $\xi \in \Gamma(E)$. Then (2.1), (2.3) straightforwardly lead to

$$
\begin{aligned}
&D_X Y = \nabla_X Y - \tfrac{1}{2}(\omega(X)Y + \omega(Y)X - g(X, Y)B), \\
&H(X, Y) = h(X, Y) + \tfrac{1}{2}g(X, Y)\mathrm{nor}(B^*), \\
&A_\xi X = a_\xi X + \tfrac{1}{2}\omega^*(\xi)X, \\
&D^\perp_X \xi = \nabla^\perp_X \xi - \tfrac{1}{2}\omega(X)\xi.
\end{aligned} \tag{2.4}
$$

Here $B = \tan(B^*)$. We establish the following:

LEMMA 1. *Let M be an invariant submanifold of H^m. The following formulae hold*:

$$a_\xi JX + Ja_\xi X = -\omega^*(\xi)JX, \tag{2.5}$$

$$a_{J\xi}X = Ja_\xi X + \tfrac{1}{2}(\omega^*(\xi)JX - \omega^*(J\xi)X), \tag{2.6}$$

$$\nabla^\perp_X J\xi = J\nabla^\perp_X \xi, \tag{2.7}$$

for any $X \in \mathscr{X}(M)$, $\xi \in \Gamma(E)$.

PROOF. Since M is invariant, by $D^*J = 0$ one has $A_\xi JX + JA_\xi X = 0$, $A_{J\xi}X = JA_\xi X$. These and (2.4) yield (2.5) and (2.6). Similarly (2.7) follows from $D^\perp_X J\xi = JD^\perp_X \xi$. Q.E.D.

LEMMA 2. *The Gauss-Codazzi-Ricci equations of a submanifold M in H^m are given by*:

$$
\begin{aligned}
R(X, Y)Z = (X \wedge Y)Z + \tfrac{1}{4}\{&[\omega(X)Y - \omega(Y)X]\omega(Z) \\
&+ [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]B\} \\
+ a_{h(Y,Z)}X - a_{h(X,Z)}Y&,
\end{aligned} \tag{2.8}
$$

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \tfrac{1}{4}(g(X, Z)\omega(Y) - g(Y, Z)\omega(X))\mathrm{nor}(B^*), \tag{2.9}$$

$$g_0(R^\perp(X, Y)\xi, \eta) - g([a_\xi, a_\eta]X, Y) = 0, \tag{2.10}$$

for all X, Y, $Z \in \mathscr{X}(M)$, ξ, $\eta \in \Gamma(E)$.

The proof of Lemma 2 follows from (2.2) and (2.3) in a straightforward manner; for instance, by (2.2) we have

$$R^*(X, Y)\xi = \tfrac{1}{4}(\omega(X)Y - \omega(Y)X)\omega^*(\xi),$$

such that $\mathrm{nor}(R^*(X, Y)\xi) = 0$.

3. Proof of Theorem 1

Let us combine (1.1) with (2.10) and put $\eta = J\xi$ in the resulting equation; this gives

$$\rho_0 \Omega(X, Y) \| \xi \|^2 - g([a_\xi, a_\eta]X, Y) = 0 \tag{3.1}$$

for any $X, Y \in \mathscr{X}(M)$, $\xi \in \Gamma(E)$. At this point, by Lemma 1, we establish

$$[a_\xi, a_{J\xi}]X = -2J(a_\xi^2 X + \omega^*(\xi) a_\xi X + \tfrac{1}{4}\omega^*(\xi)^2 X). \tag{3.2}$$

Now the substitution of (3.2) into (3.1) leads to

$$a_\xi^2 + \omega^*(\xi) a_\xi + \tfrac{1}{2}[\rho_0 \| \xi \|^2 + \tfrac{1}{2}\omega^*(\xi)^2]I = 0. \tag{3.3}$$

Let us put $m = n + p, p \geqq 1$, i.e. $n < m$, by hypothesis. Then either $p = 1$, i.e. M is a complex hypersurface of H^m, or $p > 2$. For the last case let $\{V_1, \ldots, V_p, JV_1, \ldots, JV_p\}$ be an orthonormal frame (locally defined) on E. Note that (3.3) is equivalent to

$$A_\xi^2 = -\tfrac{1}{2}\rho_0 \| \xi \|^2 I. \tag{3.4}$$

Now, on the one hand $A_{V_i}A_{V_j} + A_{V_j}A_{V_i} = 0$, $i \neq j$; on the other (by the Ricci equations (2.10) for $\xi = V_i$, $\eta = V_j$) we have $a_{V_i}a_{V_j} - a_{V_j}a_{V_i} = 0$ which yields $A_{V_i}A_{V_j} - A_{V_j}A_{V_i} = 0$, such that $A_{V_i}A_{V_j} = 0$, $i \neq j$; finally, by (3.4) we obtain $A_\xi = 0$ or

$$a_\xi = -\tfrac{1}{2}\omega^*(\xi) I, \tag{3.5}$$

that is, M is totally-umbilical, and $\rho_0 = 0$, i.e. $R^\perp = 0$.Now substitution from (3.5) into the Gauss equation (2.8) and further contraction of indices lead to

$$\mathrm{Ric}(X, Y) = \left[2(2n-1) - \frac{n}{2} \| \omega \|^2\right] g(X, Y) - \frac{n-1}{2}\, \omega(X)\omega(Y), \tag{3.6}$$

i.e., M is quasi-Einstein. Our Theorem 1 might be contrasted with a result in I. Ishihara [9].

4. Proof of Theorem 3

We define a covariant derivative of $R^\perp$ in the usual manner, cf. [15, p. 115]. Then

$$\sum_{\text{cycl.}} (\nabla_X R^\perp)(Y, Z) = 0 \tag{4.1}$$

for any $X, Y, Z \in \mathscr{X}(M)$. Here $\Sigma_{\text{cycl.}}$ denotes the cyclic sum over X, Y, Z. On the other hand

$$(4.2)\qquad (\nabla_X R^{\perp})(Y, Z)\xi = X(\rho_0)\Omega(Y, Z)J\xi + \rho_0(\nabla_X \Omega)(Y, Z)J\xi.$$

Of course, the induced connection of the invariant submanifold M is not almost-complex; in turn, we have

$$(4.3)\qquad J\nabla_X Y = \nabla_X JY + \tfrac{1}{2}(\omega(Y)JX - \theta(Y)X + \Omega(X, Y)B + g(X, Y)A)$$

where θ denotes the 1-form induced on M by the anti-Lee form $\theta^* = \omega^* \circ J$, and $A = \tan(A^*)$, $A^* = -JB^*$. Consequently, the Kaehler 2-form is not parallel any longer. Yet (4.3) yields

$$(4.4)\qquad \begin{aligned}&(\nabla_X \Omega)(Y, Z)\\ &= \tfrac{1}{2}[\theta(Z)g(X, Y) - \theta(Y)g(X, Z) + \omega(Z)\Omega(X, Y) - \omega(Y)\Omega(X, Z)].\end{aligned}$$

Finally, part (i) of Theorem 3 follows from the more general:

PROPOSITION. *Any invariant submanifold, of a l.c.K. manifold, with semi-flat normal connection for some* $\rho_0 : M \to (0, +\infty)$, *is globally conformal Kaehler provided M has complex dimension* $n \geqq 2$.

Indeed, combining (4.2), (4.4) and (4.1) we obtain

$$(4.5)\qquad \sum_{\text{cycl.}} X(\rho_0)\Omega(Y, Z) = -3\rho_0(\omega \wedge \Omega)(X, Y, Z).$$

Let us put $Z = JY$ in (4.5); since $n \geqq 2$ one may choose X orthogonal to Y, JY. Thus $d\rho_0 + \rho_0\omega = 0$. Q.E.D.

To prove the second part of Theorem 3 we suppose M is Einstein, i.e. $\text{Ric} = \lambda g$, for some $\lambda \in IR$. By (3.6) we obtain

$$(4.6)\qquad \left[2(2n-1) - \frac{n}{2}\,\|\omega\|^2 - \lambda\right]X = \frac{n-1}{2}\,\omega(X)B$$

for any $X \in \mathscr{X}(M)$. Let us put $X = B$ in (4.6) and take the inner product with B. Since M is strongly non-Kaehler (i.e. $\omega_x \neq 0$, at any $x \in M$), we obtain

$$2(2n-1) - \frac{2n-1}{2}\,\|\omega\| = \lambda;$$

substitution in (4.6) now gives $\omega(X)B = \|\omega\|^2 X$, which for $X = JB$ furnishes $\|\omega\| = 0$, a contradiction.

5. Submanifolds with non-positive Yamabe invariant

Let M^{2n} be a submanifold of H^m. By (2.8) we may compute the Ricci curvature of M^{2n}:

$$R_{jk} = (2n - 1 - \tfrac{1}{4} \| \omega \|^2) g_{jk} - \frac{n-1}{2} \omega_j \omega_k + (a_{h(\partial_j, \partial_k)} \partial_i)^i - (a_{h(\partial_i, \partial_k)} \partial_j)^i \tag{5.1}$$

where $\partial_i = \partial / \partial x^i$. Consequently, the scalar curvature ρ of M^{2n} is expressed by

$$\rho = (2n-1)(2n - \tfrac{1}{2} \| \omega \|^2) + \sum_{b=1}^{\operatorname{codim}(M^{2n})} [(\operatorname{Trace}(a_b))^2 - \operatorname{Trace}(a_b^2)]. \tag{5.2}$$

If M^{2n} is totally-umbilical then (5.1) leads to our Theorem 2, i.e. we obtain

$$\operatorname{Ric} = [2n - 1 - \tfrac{1}{4} \| \omega \|^2 + (2n-1) \| \mu \|^2] g - \frac{n-1}{2} \omega \otimes \omega \tag{5.3}$$

where μ denotes the mean curvature vector of M^{2n}; also (5.2) reduces to

$$\rho = (2n-1)(2n - \tfrac{1}{2} \| \omega \|^2) + 2n(2n-1) \| \mu \|^2. \tag{5.4}$$

Furthermore, we prove our Theorem 4. If $\mu_0 \leqq 0$ then there exists $\varphi \in C^\infty(M^{2n})$, $\varphi \geqq 0$, $\varphi \not\equiv 0$, such that $I(\varphi) \leqq 0$. Using (1.2), (5.4) we obtain

$$\frac{2}{n-1} \int_{M^{2n}} \| d\varphi \|^2 * 1 + \int_{M^{2n}} \varphi^2 (2n - \tfrac{1}{2} \| \omega \|^2) * 1$$

$$+ 2n \int_{M^{2n}} \varphi^2 \| \mu \|^2 * 1 \leqq 0$$

where $*\,1$ denotes the canonical Riemann measure on the compact submanifold M^{2n}. Since $n \geqq \frac{1}{4} \| \omega \|^2$ we obtain $d\varphi = 0$, i.e. $\varphi = \text{const}$ (we always assume M^{2n} connected). Yet $\varphi \not\equiv 0$ yields $\| \mu \| = 0$; thus $h = 0$ since M^{2n} is totally-umbilical. Moreover, we obtain ω is parallel. Consequently, if M^{2n} is invariant, then it is a l.c.K. manifold with a parallel Lee form, i.e. a g.H.m., cf. [12], [14]. To prove the last statement of Theorem 4, let K be the curvature of D, i.e. the connection induced on M^{2n} by the Weyl connection of the ambient Hopf manifold. We recall [4]:

$$
\begin{aligned}
g(K(X, Y)Z, U) = R(U, Z, X, Y) &- \tfrac{1}{2}L(X, Z)g(Y, U) + \tfrac{1}{2}L(X, U)g(Y, Z) \\
&+ \tfrac{1}{2}L(Y, Z)g(X, U) - \tfrac{1}{2}L(Y, U)g(X, Z) \\
&+ \omega^*(h(X, Z))g(Y, U) - \omega^*(h(Y, Z))g(X, U) \\
&+ \omega^*(h(Y, U))g(X, Z) - \omega^*(h(X, U))g(Y, Z) \\
&- \tfrac{1}{4}(\| \omega^* \|^2 + \| \mathrm{nor}(B^*) \|^2) \\
&\times (g(Y, Z)g(X, U) - g(X, Z)g(Y, U))
\end{aligned} \tag{5.5}
$$

for any $X, Y, Z, U \in \mathscr{X}(M^{2n})$. Here R denotes the Riemann–Christoffel tensor of M^{2n}. Also $L(X, Y) = (\nabla_X \omega)Y + \frac{1}{2}\omega(X)\omega(Y)$ and $\| \omega^* \| = 2$, cf. [12]. Let (x^i) be real-analytic local coordinates on H^m; we recall that g_0 is locally conformal to the (local) Kaehler metrics $g_0' = \delta_{ij}dx^i dx^j$, i.e. $g_0 = |x|^{-2}g_0'$, where $|x| = (\delta_{ij}x^i x^j)^{1/2}$. Let f be the restriction to M^{2n} of $2 \log |x|$. Let also k' be the sectional curvature of the metrics induced by the local Kaehler metrics g_0' on M^{2n}. By (5.5) we obtain

$$
\exp(-f)k'(p) = k(p) - 1 + \tfrac{1}{2}(\omega(X)^2 + \omega(Y)^2) \tag{5.6}
$$

for any $p \in G_2(M^{2n})$ and any g-orthonormal basis $\{X, Y\}$ in p. The condition in Theorem 4 yields $k' \geqq 0$; as $\mu_0 \leqq 0$ we may apply a result of I. Vaisman, i.e. Theorem 3 in [13, p. 281], to conclude that M^{2n} is a g.c.K. manifold.

6. Conformally flat submanifolds of a Hopf manifold

Let M^{2n} be a totally-umbilical submanifold of H^m, $n > 2$. We consider the Weyl conformal curvature tensor of M^{2n}:

$$
\begin{aligned}
W(X, Y) = R(X, Y) &- \frac{1}{2(n-1)} (QX \wedge Y + X \wedge QY) \\
&+ \frac{\rho}{4(n-1)(2n-1)} X \wedge Y
\end{aligned} \tag{6.1}
$$

for all $X, Y \in \mathscr{X}(M^{2n})$. Here $Q = \#\,\mathrm{Ric}$, or, by (5.3),

$$
QX = [(2n-1)(1 + \| \mu \|^2) - \tfrac{1}{4} \| \omega \|^2]X - \frac{n-1}{2}\, \omega(X)B. \tag{6.2}
$$

Taking into account (5.4), (6.2) the expression of the conformal tensor (6.1) becomes

$$W(X, Y) = R(X, Y) + \frac{1}{2(n-1)}\left[\frac{\|\omega\|^2}{4} - (3n-2)(1 + \|\mu\|^2)\right] X \wedge Y \tag{6.3}$$
$$+ \tfrac{1}{4}[\omega(Y)X \wedge B - \omega(X)Y \wedge B].$$

Finally, one may use the Gauss equation (2.8) and

$$a_{h(Y,\cdot)}X - a_{h(X,\cdot)}Y = \|\mu\|^2 X \wedge Y;$$

consequently (6.3) reduces to

$$W(X, Y) = \frac{1}{2(n-1)}\left[\frac{\|\omega\|^2}{4} - n(1 + \|\mu\|^2)\right] \cdot X \wedge Y \tag{6.4}$$

for any $X, Y \in \mathscr{X}(M^{2n})$. Now $W = 0$ iff

$$\|\omega\|^2 = 4n(1 + \|\mu\|^2), \tag{6.5}$$

i.e. iff $\rho = 0$, cf. our (5.4). This proves Theorem 5. As remarked is §1, Theorem 2 in [7, p. 234] might not be applied to our case since (5.3) yields

$$\frac{\|\mathrm{Ric}\|}{1 + \|\mu\|^2} = (n-1)[2n(2n-1)]^{1/2}. \tag{6.6}$$

To prove Theorem 6, by (2.8) we obtain the expression of the Riemann–Christoffel tensor of the (totally-umbilical) submanifold M^{2n}, i.e.

$$R(W, Z, X, Y) = (1 + \|\mu\|^2)(g(Y, Z)g(X, W) - g(X, Z)g(Y, W))$$
$$+ \tfrac{1}{4}\{[\omega(X)g(Y, W) - \omega(Y)g(X, W)]\omega(Z)$$
$$+ [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]\omega(W)\}$$

which yields the sectional curvature of M^{2n}:

$$k(p) = 1 + \|\mu\|^2 - \tfrac{1}{4}[\omega(X)^2 + \omega(Y)^2], \qquad \text{for any } p \in G_2(M^{2n}),$$

spanned by the orthonormal vectors X, Y. Suppose now that M^{2n} is a space-form, i.e. $k(p) = c$, $c \in IR$, for all $p \in G_2(M^{2n})$. Then $\rho = 2n(2n-1)c$; thus $W = 0$ yields $c = 0$ or $4(1 + \|\mu\|^2) = \omega(X)^2 + \omega(Y)^2$; let $Y = \|B\|^{-1}B$. Then $n = 0$ by (6.5), a contradiction.

7. Totally-umbilical submanifolds of zero scalar curvature

Let M^{2n} be an invariant submanifold of H^m, $\rho = 0$, $h = g \otimes \mu$. By a theorem in [4] for each invariant submanifold in a l.c.K. manifold the mean curvature vector is given by $\mu = -\frac{1}{2}\,\mathrm{nor}(B^*)$. Also $\|\omega^*\|^2 = \|\omega\|^2 + \|\mathrm{nor}(B^*)\|^2$, $\|\omega^*\| = 2$, yield (by (6.5)) $\|\omega\|^2 = 8n/(n+1)$; but $4 - 8n/(n+1) \geqq 0$ yields $n = 1$ and $\mu = 0$, such that M^2 is a minimal surface in H^m; thus $h = 0$ and by a theorem in [4], M^2 is flat.

To establish Theorem 8, let

$$P(X, Y) = R(X, Y) - \frac{1}{n-1}(X \wedge Y) \circ Q \tag{7.1}$$

be the projective curvature tensor of M^{2n}, $n \geqq 2$. By (6.2), (2.8), we put (7.1) in the following form:

$$\begin{aligned} P(X, Y)Z = \frac{1}{n-1} [\tfrac{1}{4}\|\omega\|^2 - n(1 + \|\mu\|^2)](X \wedge Y)Z \\ + \tfrac{1}{4}\{[\omega(Y)X - \omega(X)Y]\omega(Z) \\ + [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]B\}. \end{aligned} \tag{7.2}$$

If $\rho = 0$ then by (6.5) and $P = 0$ we obtain

$$(\omega(Y)X - \omega(X)Y)\omega(Z) = (g(Y, Z)\omega(X) - g(X, Z)\omega(Y))B. \tag{7.3}$$

We put $Z = B$ in (7.3); $\|\omega\| \neq 0$ everywhere gives $\omega(Y)X - \omega(X)Y = 0$ which, together with (7.3), gives

$$g(Y, Z) = \frac{1}{\|\omega\|^2}\,\omega(Y)\omega(X),$$

and in particular $\omega(Y) = \|\omega\|\ \|Y\|$. This yields $\|Y\|X - \|X\|Y = 0$ for any X, Y, a contradiction.

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